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Normal families and uniqueness theorems for entire functions

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Abstract

There exists a set S with 3 elements such that if f is a non-constant entire function satisfying $E(S,f)=E(S,f')$, then $f\equiv f'$. The number 3 is best possible. The proof uses the theory of normal families in an essential way.
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1. Introduction

Let f be a non-constant meromorphic function in the complex plane and let S be a set of complex numbers. Put

$$E(S,f)=\bigcup_{a\in S}\{z\colon\ f(z)-a=0\},$$

where a zero of multiplicity m is counted m times in the set.

Answering a question of Gross [2], Yi [14] proved the following result.

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Theorem A. *There exists a finite set S containing 7 elements such that if f and g are two non-constant entire functions and $E(S, f) = E(S, g)$, then $f \equiv g$.*

Earlier, Rubel and Yang [10] had shown

Theorem B. *Let a and b be distinct complex numbers, and let f be a non-constant entire function. If $E(a, f) = E(a, f')$ and $E(b, f) = E(b, f')$, then $f \equiv f'$.*

In this paper, we use the theory of normal families to prove

Theorem 1. *There exists a set S with 3 elements such that if a non-constant entire function f and its derivative f' satisfy $E(S, f) = E(S, f')$, then $f \equiv f'$.*

Let $S = \{a, b\}$, where a and b are any two distinct complex numbers. Let $f(z) = e^{-z} + a + b$; then $f'(z) = -e^{-z}$. Obviously, $E(S, f) = E(S, f')$, but $f \not\equiv f'$. This shows that the number 3 in Theorem 1 is best possible.

Jank et al. [6] proved

Theorem C. *Let f be a non-constant entire function, and let a be a non-zero constant. If $E(a, f) = E(a, f')$ and $f''(z) = a$ whenever $f'(z) = a$, then $f \equiv f'$.*

Again, using the theory of normal families, we prove

Theorem 2. *Let f be a non-constant entire function and $k \geqslant 2$ a positive integer. Let a and b be complex numbers such that $b \neq 0$. If $E(a, f) = E(a, f')$ and $f^{(k)}(z) = b$ whenever $f'(z) = b$, then*

$$f(z) = de^{cz} + \frac{c-1}{c}a,$$

where c and d are two non-zero constants with $c^{k-1} = 1$. In particular, $f \equiv f'$ for $k = 2$.

From Theorem 2 we obtain the following result.

Theorem D [7, Theorem 2]. *Let f be a non-constant entire function and $k \geqslant 2$ a positive integer. Let a be a non-zero constant. If $E(a, f) = E(a, f') = E(a, f^{(k)})$, then*

$$f(z) = de^{cz} + \frac{c-1}{c}a,$$

where c, d are two non-zero constants with $c^{k-1} = 1$.

It does not seem that Theorem 2 can be proved by using the methods in [6] and [7].

Gundersen [3] and Yang [13] proved

Theorem E. *Let a be a non-zero complex number and k a positive integer. Let f be a non-constant entire function. If $f(z)f^{(k)}(z) \neq 0$ and $f(z) = a$ if and only if $f^{(k)}(z) = a$, then $f(z) = e^{Az+B}$, where $A \neq 0$ and B are constants satisfying $A^k = 1$.*

As an application of the theory of normal families, we improve Theorem E as follows.

Theorem 3. *Let a and b be distinct non-zero complex numbers and k a positive integer. Let f be a non-constant entire function. If $f(z) \neq 0$ and $f^{(k)}(z) = b$ whenever $f(z) = a$, then $f(z) = e^{Az+B}$, where $A \neq 0$ and B are constants satisfying $A^k = b/a$.*

Corollary 4. *Let a be a non-zero complex number and k a positive integer. Let f be a non-constant entire function. If $f(z) \neq 0$ and $f^{(k)}(z) = a$ whenever $f(z) = a$, then $f(z) = e^{Az+B}$, where $A \neq 0$ and B are constants satisfying $A^k = 1$.*

Theorem 5. *Let a and b be distinct non-zero complex numbers and k a positive integer. Let f be a non-constant entire function. If $f(z) \neq 0$ and $f(z) = a$ whenever $f^{(k)}(z) = b$, then $f(z) = e^{Az+B}$, where $A \neq 0$ and B are constants satisfying $A^k = b/a$.*

Corollary 6. *Let a be a non-zero complex number and k a positive integer. Let f be a non-constant entire function. If $f(z) \neq 0$ and $f(z) = a$ whenever $f^{(k)}(z) = a$, then $f(z) = e^{Az+B}$, where $A \neq 0$ and B are constants satisfying $A^k = 1$.*

Throughout this paper, we use the standard notation of Nevanlinna theory (cf. [5,12]). In particular, $S(r, f)$ denotes any function satisfying

$$S(r, f) = O\big(\log T(r, f)\big) + O(\log r)$$

as $r \to +\infty$, possibly outside of a set of positive measure, where $T(r, f)$ is Nevanlinna's characteristic function. In fact, the functions for which we use this notation are all of finite order, so the exceptional set does not occur. For such functions, we have $S(r, f) = o(T(r, f))$ (cf. [5, p. 41]).

2. Some lemmas

For the proof of our results, we need the following lemmas.

Lemma 1 ([1], cf. [8]). *Let f be an entire function, M a positive number. If $f^{\#}(z) \leqslant M$ for any $z \in \mathbb{C}$, then f is of exponential type.*

Here, as usual, $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ is the spherical derivative.

Lemma 2 [4, Theorem 1]. *Let f be a non-constant entire function with finite order, and let a be a finite value. If $E(a, f) = E(a, f')$, then*

$$f'(z) - a = A\big[f(z) - a\big],$$

where A is a non-zero constant.

Lemma 3 [9, Lemma 2]. *Let $\mathcal{F}$ be a family of functions holomorphic on the unit disc, all of whose zeros have multiplicity at least k. Suppose that there exists $A \geqslant 1$ such that $|f^{(k)}(z)| \leqslant A$ whenever $f(z) = 0$. If $\mathcal{F}$ is not normal, there exist, for each $0 \leqslant \alpha \leqslant k$,*

(a) *a number* $0 < r < 1$,
(b) *points* z_n *with* $|z_n| < r$,
(c) *functions* $f_n \in \mathcal{F}$, *and*
(d) *positive numbers* $\rho_n \to 0$

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \to g(\zeta)$ *locally uniformly, where* g *is a non-constant entire function, all of whose zeros have multiplicity at least* k, *such that* $g^{\#}(\zeta) \leqslant g^{\#}(0) = kA + 1$. *In particular,* g *is of exponential type.*

For $0 \leqslant \alpha < k$, the hypothesis on $f^{(k)}(z)$ can be dropped, and $kA + 1$ can be replaced by an arbitrary positive constant.

Lemma 4. *Let* $\mathcal{F}$ *be a family of holomorphic functions in a domain* D. *Let* k *be a positive integer. Let* a, b, *and* c *be three distinct finite complex numbers and* M *a positive number. If, for any* $f \in \mathcal{F}$, *the zeros of* f *are of multiplicity* $\geqslant k$ *and* $|f^{(k)}(z)| \leqslant M$ *whenever* $f(z) = a, b$, *or* c, *then* $\mathcal{F}$ *is normal in* D.

Proof. Suppose that $\mathcal{F}$ is not normal on D. By Lemma 3, there exist points $z_n \in D$, positive numbers $\rho_n \to 0^+$, and functions $f_n \in \mathcal{F}$ such that $g_n(\xi) = f_n(z_n + \rho_n \xi)$ converges locally uniformly to a non-constant entire function g, whose zeros have multiplicity $\geqslant k$.

Obviously, $g^{(k)}(\xi) \not\equiv 0$, for otherwise g would be a polynomial of degree less than k, and so could not have zeros of multiplicity at least k.

We claim that $g^{(k)}(\xi) = 0$ whenever $g(\xi) = a, b, c$.

Suppose that $g(\xi_0) = a$. Then by Hurwitz's theorem, there exists a sequence $\{\xi_n\}$ with $\xi_n \to \xi_0$ such that (for n large) $a = g_n(\xi_n) = f_n(z_n + \rho_n \xi_n)$. Thus $|f_n^{(k)}(z_n + \rho_n \xi_n)| \leqslant M$. Hence $|g_n^{(k)}(\xi_n)| = |\rho_n^k f_n^{(k)}(z_n + \rho_n \xi_n)| \leqslant \rho_n^k M$. Since $g^{(k)}(\xi_0) = \lim_{n\to\infty} g_n^{(k)}(\xi_n) = 0$, we have shown that $g^{(k)}(\xi) = 0$ whenever $g(\xi) = a$. Likewise, if $g(\xi_0) = b$ or c, then $g^{(k)}(\xi_0) = 0$.

Using standard results of Nevanlinna theory, we have

$$\begin{aligned} 2T(r, g) &\leqslant \overline{N}\left(r, \frac{1}{g-a}\right) + \overline{N}\left(r, \frac{1}{g-b}\right) + \overline{N}\left(r, \frac{1}{g-c}\right) + S(r, g) \\ &\leqslant N\left(r, \frac{1}{g^{(k)}}\right) + S(r, g) \leqslant T\left(r, \frac{1}{g^{(k)}}\right) + S(r, g) \\ &\leqslant T(r, g^{(k)}) + S(r, g) \leqslant T(r, g) + S(r, g). \end{aligned}$$

Note that we have used the fact that g is entire in both the first and last inequalities above.

Thus we get a contradiction: $T(r, g) = o(T(r, g))$. Hence $\mathcal{F}$ is normal in D. This completes the proof of the lemma. □

Example. Let $S = \{1, -1\}$. Set

$$\mathcal{F} = \{f_n(z)\colon n = 2, 3, 4, \ldots\},$$

where

$$f_n(z) = \frac{n+1}{2n} e^{nz} + \frac{n-1}{2n} e^{-nz}, \qquad D = \{z\colon |z| < 1\}.$$

Then, for any $f_n \in \mathcal{F}$, we have

$$n^2\left[f_n^2(z) - 1\right] = \left[f_n'(z)\right]^2 - 1.$$

Thus $E(S, f) = E(S, f')$, but $\mathcal{F}$ is not normal in D.

Lemma 5. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D and $k \geqslant 2$ a positive integer. Let a, b, and c be three complex numbers such that $b \neq 0$. If, for any $f \in \mathcal{F}$, $E(0, f) = E(a, f')$ and $f^{(k)}(z) = c$ whenever $f'(z) = b$, then $\mathcal{F}$ is normal in D.*

Proof. Suppose that $\mathcal{F}$ is not normal on D. By Lemma 3, there exist sequences $z_n \in D$, $\rho_n \to 0^+$, and $f_n \in \mathcal{F}$ such that $g_n(\xi) = \rho_n^{-1} f_n(z_n + \rho_n \xi)$ converges locally uniformly to a non-constant entire function g of exponential type.

We consider two cases.

Case 1: $a \neq 0$. Suppose that $g(\xi_0) = 0$. Then by Hurwitz's theorem, there exists a sequence $\{\xi_n\}$ with $\xi_n \to \xi_0$ such that (for n sufficiently large)

$$g_n(\xi_n) = \rho_n^{-1} f_n(z_n + \rho_n \xi_n) = 0.$$

Thus $f_n(z_n + \rho_n \xi_n) = 0$. Since $E(0, f_n) = E(a, f_n')$, we have

$$g_n'(\xi_n) = f_n'(z_n + \rho_n \xi_n) = a.$$

Hence $g'(\xi_0) = \lim_{n\to\infty} g_n'(\xi_n) = a$. Thus $g'(\xi) = a$ whenever $g(\xi) = 0$.

Now suppose that $g'(\xi_0) = a$. We claim that $g'(\xi) \not\equiv a$, for otherwise $g(\xi) = a(\xi - \xi_1)$. A simple calculation then shows that

$$g^{\#}(0) \leqslant \begin{cases} 1 & \text{if } |\xi_1| \geqslant 1, \\ |a| & \text{if } |\xi_1| < 1. \end{cases}$$

Hence we have $g^{\#}(0) < (|a| + 1) + 1$, which is a contradiction. Since $g'(\xi_0) = a$ but $g'(\xi) \not\equiv a$, there exist ξ_n, $\xi_n \to \xi_0$, such that (for n large) $f_n'(z_n + \rho_n \xi_n) = g_n'(\xi_n) = a$. It follows that $f_n(z_n + \rho_n \xi_n) = 0$, so that $g_n(\xi_n) = f_n(z_n + \rho_n \xi_n)/\rho_n = 0$. Since $g(\xi_0) = \lim_{n\to\infty} g_n(\xi_n) = 0$, we have shown that $g(\xi) = 0$ whenever $g'(\xi) = a$.

Thus $g(\xi) = 0$ if and only if $g'(\xi) = a$.

Let ξ_0 be a zero of $g'(\xi) - a$ with multiplicity $m \geqslant 1$. Then $g(\xi_0) = 0$, and there exists a positive number $\delta > 0$ such that for $0 \leqslant |\xi - \xi_0| < \delta$

$$g'(\xi) \neq 0. \tag{2.1}$$

By Hurwitz's theorem, there exist m sequences $\{\xi_{in}\}$ $(i = 1, 2, \ldots, m)$ such that $\lim_{n\to\infty} \xi_{in} = \xi_0$, and (for large n)

$$g_n'(\xi_{in}) = a, \quad i = 1, 2, \ldots, m. \tag{2.2}$$

Thus

$$f_n'(z_n + \rho_n \xi_{in}) = a, \quad i = 1, 2, \ldots, m.$$

Hence, by $E(0, f_n) = E(a, f_n')$ and $a \neq 0$, we have $f_n(z_n + \rho_n \xi_{in}) = 0$ for $i = 1, 2, \ldots, m$, and each ξ_{in} is a simple zero of g_n. Thus

$$\xi_{in} \neq \xi_{jn}, \quad 1 \leqslant i < j \leqslant m. \tag{2.3}$$

By Rouché's theorem, the order of the zero of g at ξ_0 is m. This implies that $E(0, g) = E(a, g')$. (In fact, $m = 1$.)

If $g' \neq b$, then since g is of order at most one, there exist non-zero constants A and B such that

$$g'(\xi) = Be^{A\xi} + b. \tag{2.4}$$

Thus we have

$$g(\xi) = \frac{B}{A}e^{A\xi} + b\xi + C. \tag{2.5}$$

Obviously, $g(\xi) = 0$ has infinitely many solutions. Suppose $g(\xi_0) = 0$. Then by (2.4), (2.5), and $E(0, g) = E(a, g')$, we get $\xi_0 = (b - CA - a)/bA$, which is a contradiction.

Thus there exists ξ_0 such that $g'(\xi_0) = b$. Clearly, $g'(\xi) \not\equiv b$, for otherwise $g(\xi) = b\xi + C$, which contradicts $E(0, g) = E(a, g')$. Hence there exist ξ_n, $\xi_n \to \xi_0$, such that (for n large) $g_n'(\xi_n) = b$. Thus, $f_n'(z_n + \rho_n\xi_n) = b$. Since $f_n^{(k)} = c$ whenever $f_n' = b$, we have

$$f_n^{(k)}(z_n + \rho_n\xi_n) = c \quad \text{and} \quad g_n^{(k)}(\xi_n) = \rho_n^{k-1} f_n^{(k)}(z_n + \rho_n\xi_n) = \rho_n^{k-1}c \to 0$$

as $n \to \infty$. Thus $g^{(k)}(\xi_0) = \lim_{n\to\infty} g_n^{(k)}(\xi_n) = 0$. By Lemma 2, we have

$$g'(\xi) - a = Ag(\xi),$$

where A is a non-zero constant.

Thus we have

$$g(\xi) = Be^{A\xi} - \frac{a}{A}, \tag{2.6}$$

$$g^{(k)}(\xi) = A^k Be^{A\xi}, \tag{2.7}$$

where B is a non-zero constant.

By $g^{(k)}(\xi_0) = 0$, (2.7), and $AB \neq 0$, we have a contradiction.

Case 2: $a = 0$. In this case, it is clear that $g(\xi) \neq 0$. Thus

$$g(\xi) = Be^{A\xi}, \tag{2.8}$$

where A, B are non-zero constants. Clearly, there exists ξ_0 such that $g'(\xi_0) = b$. Using the same argument as in Case 1, we obtain $g^{(k)}(\xi_0) = 0$, which contradicts (2.8).

Hence $\mathcal{F}$ is normal in D. This completes the proof of the lemma. □

Lemma 6. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D and k a positive integer. Let a and b be distinct non-zero complex numbers. If, for any $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)}(z) = b$ whenever $f(z) = a$, then $\mathcal{F}$ is normal in D.*

Proof. Suppose that $\mathcal{F}$ is not normal on D. By Lemma 3, there exist points $z_n \in D$, numbers $\rho_n \to 0^+$, and functions $f_n \in \mathcal{F}$ such that $g_n(\xi) = f_n(z_n + \rho_n\xi)$ converges locally uniformly to a non-constant entire function g. Moreover, g has no zeros and is of exponential type. It follows that $g(\xi) = e^{A\xi+B}$, where $A \neq 0$ and B are constants. Suppose that $g(\xi_0)=a$. Then by Hurwitz's theorem, there exist ξ_n, $\xi_n\to\xi_0$, such that (for n large) $a=g_n(\xi_n)=f_n(z_n+\rho_n\xi_n)$. Hence $f^{(k)}(z_n+\rho_n\xi_n)=b$, so that

$$g^{(k)}(\xi_0)=\lim_{n\to\infty}g_n^{(k)}(\xi_n)=\lim_{n\to\infty}\rho_n^k f_n^{(k)}(z_n+\rho_n\xi_n)=\lim_{n\to\infty}\rho_n^k b=0.$$

This is a contradiction, since $g^{(k)}(\xi_0)=A^k e^{A\xi_0+B}\neq 0$. The proof of the lemma is completed. □

Using the same argument as in the proof of Lemma 6, we can prove the following lemma. We omit the details here.

Lemma 7. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, let k be a positive integer, and let a, b be two non-zero finite complex numbers. If, for any $f\in\mathcal{F}$, $f\neq 0$ and $f(z)=a$ whenever $f^{(k)}(z)=b$, then $\mathcal{F}$ is normal in D.*

Finally, we recall Marty's well-known characterization of normal families.

Lemma 8 [11, p. 75]. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. Then $\mathcal{F}$ is normal in D if and only if the spherical derivatives of functions $f\in\mathcal{F}$ are uniformly bounded on compact subsets of D.*

3. Proof of Theorem 1

Set $S=\{0,a,b\}$, where a, b are two non-zero distinct finite complex numbers satisfying $a^2\neq b^2$, $a\neq 2b$, $b\neq 2a$, $a^2-ab+b^2\neq 0$. Suppose that $E(S,f)=E(S,f')$. Set

$$\phi(z)=\frac{f'(z)[f'(z)-a][f'(z)-b]}{f(z)[f(z)-a][f(z)-b]}. \tag{3.1}$$

Then by $E(S,f)=E(S,f')$, there exists an entire function h satisfying

$$\phi(z)=\frac{f'(z)[f'(z)-a][f'(z)-b]}{f(z)[f(z)-a][f(z)-b]}=e^{h(z)}. \tag{3.2}$$

Standard computations involving the lemma on the logarithmic derivative (see [6, pp. 32, 34, 55]) show that

$$m(r,\phi)=S(r,f), \tag{3.3}$$

and hence

$$T(r,\phi)=S(r,f). \tag{3.4}$$

Let us now show that f is of exponential type. Set $\mathcal{F}=\{f(z+w)\colon w\in\mathbb{C}\}$. Then $\mathcal{F}$ is a family of holomorphic functions on the unit disc Δ. By the assumption, for any function $g(z)=f(z+w)$, we have $|g'(z)|\leqslant\max\{|a|,|b|\}$ whenever $g(z)=0,a,b$. Hence by Lemma 4, $\mathcal{F}$ is normal in Δ. Thus by Lemma 8, there exists $M>0$ satisfying $f^{\#}(z)\leqslant M$ for all $z\in\mathbb{C}$. By Lemma 1, f is of exponential type.

Therefore, $T(r,f)=O(r)$, whence $S(r,f)=O(\log r)$. It then follows from (3.4) that ϕ is a polynomial, so by (3.2) ϕ must be a non-zero constant A. Hence

$$\frac{f'(z)[f'(z)-a][f'(z)-b]}{f(z)[f(z)-a][f(z)-b]}=A,$$

that is,

$$f'(z)\big[f'(z)-a\big]\big[f'(z)-b\big]=Af(z)\big[f(z)-a\big]\big[f(z)-b\big]. \tag{3.5}$$

Differentiating the two sides of (3.5), we obtain

$$\big[3(f')^2-2(a+b)f'+ab\big]f''=A\big[3f^2-2(a+b)f+ab\big]f'. \tag{3.6}$$

We claim $f'\neq 0$. Indeed, suppose that $f'(z_0)=0$ and

$$f(z)=f(z_0)+A_n(z-z_0)^n+\cdots,$$

where $A_n\neq 0$, $n\geqslant 2$. Then the left-hand side of (3.6) vanishes at z_0 to order $n-2$, while the right-hand side vanishes to the order at least $n-1$, a contradiction. Hence

$$f'(z)=BCe^{Cz} \tag{3.7}$$

and

$$f(z)=D+Be^{Cz}, \tag{3.8}$$

where $B\neq 0$, $C\neq 0$, and D are constants.

If $D\neq 0,a,b$, then by Nevanlinna's second fundamental theorem,

$$T(r,f)\leqslant \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-D}\right)+S(r,f)=\overline{N}\left(r,\frac{1}{f}\right)+S(r,f),$$

that is,

$$\overline{N}\left(r,\frac{1}{f}\right)=T(r,f)+S(r,f). \tag{3.9}$$

Similarly, we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f-a}\right)&=T(r,f)+S(r,f),\\ \overline{N}\left(r,\frac{1}{f-b}\right)&=T(r,f)+S(r,f).\end{aligned} \tag{3.10}$$

By (3.9), (3.10), $E(S,f)=E(S,f')$, and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned}3T(r,f)&=\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+S(r,f)\\ &\leqslant \overline{N}\left(r,\frac{1}{f'-a}\right)+\overline{N}\left(r,\frac{1}{f'-b}\right)+S(r,f)\\ &\leqslant 2T(r,f')+S(r,f)\leqslant 2T(r,f)+S(r,f).\end{aligned}$$

Hence we obtain $T(r,f)=S(r,f)$, which contradicts (3.8). Thus $D\in\{0,a,b\}$.

Now we consider the following three cases.

Case 1: $D = 0$. By (3.7) and (3.8), we have

$$f(z) = Be^{Cz}, \qquad f'(z) = BCe^{Cz}. \tag{3.11}$$

Suppose $f(z_1) = a$. Then since $E(S, f) = E(S, f')$, we have either $f'(z_1) = a$ or $f'(z_1) = b$. If $f'(z_1) = a$, then by (3.11), $C = 1$, so $f \equiv f'$. If $f'(z_1) = b$, then by (3.11),

$$C = \frac{b}{a}. \tag{3.12}$$

Similarly, if $f(z_2) = b$, then either $f'(z_2) = a$ or $f'(z_2) = b$. If $f'(z_2) = b$, then $C = 1$, so that $f \equiv f'$. If $f'(z_2) = a$, then by (3.11),

$$C = \frac{a}{b}. \tag{3.13}$$

Thus either $f \equiv f'$ or, by (3.12) and (3.13), $a^2 = b^2$. However, this last relation is ruled out by our choice of a and b. It follows that if $D = 0$, then $f \equiv f'$.

Case 2: $D = a$. By (3.7) and (3.8), we have

$$f(z) = a + Be^{Cz}, \qquad f'(z) = BCe^{Cz}. \tag{3.14}$$

Let $f(z_3) = 0$. Then since $E(S, f) = E(S, f')$, either $f'(z_3) = a$ or $f'(z_3) = b$.

Assume first that $f'(z_3) = a$. Then by (3.14), $C = -1$. Thus

$$f(z) = a + Be^{-z}, \qquad f'(z) = -Be^{-z}. \tag{3.15}$$

Let $f(z_4) = b$. Then since $E(S, f) = E(S, f')$, either $f'(z_4) = a$ or $f'(z_4) = b$. If $f'(z_4) = a$, (3.15) gives $b = 0$, which contradicts our choice of b. If $f'(z_4) = b$, we obtain $a = 2b$, which also contradicts our choice of a and b.

A similar argument applies in case $f'(z_3) = b$. In that case, $C = -b/a$ and

$$f(z) = a + Be^{-(b/a)z}, \qquad f'(z) = -\frac{b}{a}Be^{-(b/a)z}. \tag{3.16}$$

Choosing z_4 so that $f(z_4) = b$, we have either $f'(z_4) = a$ or $f'(z_4) = b$. If $f'(z_4) = a$, (3.16) yields $a^2 - ab + b^2 = 0$, which contradicts our choice of a and b. Similarly, $f'(z_4) = b$ leads to $b = 0$, which is also ruled out.

It follows that Case 2 cannot occur.

Case 3: $D = b$. This case is symmetric to Case 2 and can be eliminated by the same arguments.

In the above discussion we have shown that $f \equiv f'$. This completes the proof of the theorem. $\square$

4. Proof of Theorem 2

First, we prove that the order of f is at most 1. Set $\mathcal{F} = \{f(z + w) - a\colon w \in \mathbb{C}\}$. Then $\mathcal{F}$ is a family of holomorphic functions on the unit disc Δ. By assumption, for any function $g(z) = f(z + w) - a$, we have that $E(0, g) = E(a, g')$ and $g^{(k)}(z) = b$ whenever $g'(z) = b$. Hence by Lemma 5, $\mathcal{F}$ is normal in Δ. Thus by Lemma 8, there exists $M > 0$

satisfying $f^{\#}(z) \leqslant M$ for all $z \in \mathbb{C}$. By Lemma 1, f is of exponential type and hence of order at most one. Thus, by Lemma 2, we have

$$f'(z) - a = c\big[f(z) - a\big], \tag{4.1}$$

where c is a non-zero constant.

Hence

$$f(z) = de^{cz} + \frac{c-1}{c}a, \tag{4.2}$$

$$f^{(k)}(z) = c^k de^{cz}, \tag{4.3}$$

where d is a non-zero constant. Clearly, there exists z_0 such that $f'(z_0) = b$. Then $f^{(k)}(z_0) = b$, so by (4.3) $c^{k-1} = 1$. This completes the proof of Theorem 2. □

5. Proofs of Theorems 3 and 5

Because the proofs of Theorems 3 and 5 are similar, we give only the proof of Theorem 3.

First, we prove that f is of exponential type. Set

$$\mathcal{F} = \big\{f(z+w)\colon\ w \in \mathbb{C}\big\}, \quad z \in D = \big\{z\colon\ |z| < 1\big\}.$$

Then $\mathcal{F}$ is a family of holomorphic functions in D. By assumption, for any function $g(z) = f(z+w)$, $g(z) \neq 0$ and $g^{(k)}(z) = b$ whenever $g(z) = a$. Hence by Lemma 6, $\mathcal{F}$ is normal in D. Thus by Lemma 8, there exists $M > 0$ satisfying $f^{\#}(z) \leqslant M$ for all $z \in \mathbb{C}$. By Lemma 1, f is of exponential type.

Since $f \neq 0$ and f is non-constant, $f(z) = e^{Az+B}$, where A $(\neq 0)$, B are constants. From $f^{(k)}(z) = b$ whenever $f(z) = a$, we obtain $A^k = b/a$. This concludes the proof of Theorem 3. □

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